ON THE PROBLEM OF AN INTENSE EXPLOSION AT THE BOUNDARY OF A HALF-SPACE CONTAINING A PERFECT GAS

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The axisymmetric problem of an intense point-source explosion [1] at the boundary of a half-space filled with a weightless perfect gas with the adiabatic exponent γ is considered. At the initial instant $t = 0$ a finite energy E_0 is released at a certain point of this boundary, i.e. a point source explosion occurs. This problem could be of interest in connection with the investigation of motion resulting from a strong shock at the medium surface when the kinetic energy of the body is sufficiently great. This problem was considered by a number of authors. Papers [2-4] dealt with this subject from the point of view of its application to the problem of crater formation when a body moving at a high space velocity hits the flat surface of another body. It was assumed there that the shock wave propagates in the same way as in the case of explosion in a boundless medium. In [5] the medium is considered to be an incompressible fluid, and that the momentum of the substance affected by the motion is time-independent and equal to the striking body momentum. In [6] certain approximate characteristics are obtained by means of construction of an exact particular solution of differential equations defining the flow. An approximate analysis of energy distribution between two media resulting from a point source explosion at their interface is given in [7 and 8]. In [9] a numerical solution of the problem of explosion on the surface of a copper plate obtained by using elasto-plastic models for the copper plate and a perfect fluid. Results of experiments with point source explosions at the surface of water are given in papers [10 and 11]. In paper [12] this problem is solved in a linearized formulation. The solution was derived in the form of Fourier series expansion in variable θ, the angle with the axis of symmetry, with coefficients dependent on a self-similar space-temporal variable. The known solution of Sedov of the problem of an intense point source explosion [1] was used for the zero terms of the expansion. Considerable space is devoted in paper [12] to the analysis of shock wave interaction with the free surface at the intersection point of these. Significant inaccuracies were, however, introduced which led to qualitative and quantitative distortions of the flow pattern as a whole. These inaccuracies have been eliminated in this paper.

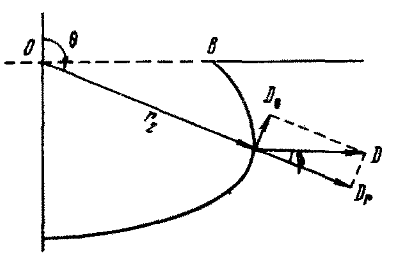

Fig. 1

In the problem here considered we may take r, θ, t, E_0, ρ_1, γ as the controlling parameters, where ρ_1 is the medium initial density, r and θ the variables of a spherical coordinate system with its origin at the undisturbed free surface, and angle θ read off from the axis of symmetry.

All of the flow characteristics depend on the following dimensionless parameters [1]:

$$\lambda = \frac{r}{E^{1/5} \rho_1^{-1/5} t^{2/5}}, \ \theta, \ \gamma \qquad (E_0 = \alpha E, \ \alpha = \alpha(\gamma))$$

In these variables the basic equations are of the form

$$R\left[\lambda\frac{\partial U}{\partial\lambda}\left(U-\frac{2}{5}\right)+V\frac{\partial U}{\partial\theta}+U^2-U-V^2\right]+\lambda\frac{\partial P}{\partial\lambda}+2P=0$$
$$R\left[\lambda\frac{\partial V}{\partial\lambda}\left(U-\frac{2}{5}\right)+V\frac{\partial V}{\partial\theta}+2UV-V\right]+\frac{\partial P}{\partial\theta}=0$$
$$\lambda\frac{\partial R}{\partial\lambda}\left(U-\frac{2}{5}\right)+V\frac{\partial R}{\partial\theta}+R\left(\lambda\frac{\partial U}{\partial\lambda}+3U+\frac{\partial V}{\partial\theta}+V\operatorname{ctg}\theta\right)=0 \qquad (1)$$
$$\lambda\frac{\partial P}{\partial\lambda}\left(U-\frac{2}{5}\right)+V\frac{\partial P}{\partial\theta}+2P(U-1)-\frac{\gamma P}{R}\left[\lambda\frac{\partial R}{\partial\lambda}\left(U-\frac{2}{5}\right)+V\frac{\partial R}{\partial\theta}\right]=0$$
$$U=\frac{t}{r}u,\ V=\frac{t}{r}v,\ R=\frac{\rho}{\rho_1},\quad P=\frac{t^2}{\rho_1 r^2}p$$

where U and V are the dimensionless values of the radial and transversal velocity components u and v, and R and P the dimensionless values of density ρ and pressure p, respectively.

Boundary conditions at the shock wave surface

$$r_2(t,\theta)=\left(\frac{E}{\rho_1}\right)^{1/5}t^{2/5}\lambda_2(\theta) \qquad (2)$$

are of the form

$$u_2=\frac{2}{\gamma+1}D_r,\quad v_2=\frac{2}{\gamma+1}D_\theta,\quad \rho_2=\frac{\gamma+1}{\gamma-1}\rho_1,\quad p_2=\frac{2}{\gamma+1}\rho_1 D^2 \qquad (3)$$

$$D=\frac{\partial r_2}{\partial t}\left[1+\left(\frac{1}{r_2}\frac{\partial r_2}{\partial\theta}\right)^2\right]^{-1/2}=\frac{2}{5}\frac{r_2}{t}\left[1+\left(\frac{\lambda_2'}{\lambda_2}\right)^2\right]^{-1/2}=\frac{2}{5}\frac{r_2}{t}\cos\beta \qquad (4)$$

Here D is the shock wave normal velocity, D_r and D_θ the components of this velocity in the directions of r and θ. Angle β is defined on Fig. 1, subscript 2 denotes magnitudes at the shock front, and a prime denotes here and in the following differentiation with respect to θ.

From the relation $D=r_2D^\circ/t$ we ontain for the shock wave dimensionless velocity the expression:

$$D^\circ={}^2/_5\cos\beta$$

In particular when $\theta={}^1/_2\pi$, angle β is equal to angle δ between the shock wave and the normal to the undisturbed free surface. An incorrect expression was used in [12] for the dimensionless normal velocity of the shock wave for $\theta={}^1/_2\pi$.

The conditions of the shock wave may be written in the following dimensionless form:

$$U_2(\theta)=\frac{4}{5(\gamma+1)}\left[1+\left(\frac{\lambda_2'}{\lambda_2}\right)^2\right]^{-1},\qquad V_2(\theta)=-\frac{4}{5(\gamma+1)}\frac{\lambda_2'}{\lambda_2}\left[1+\left(\frac{\lambda_2'}{\lambda_2}\right)^2\right]^{-1}$$
$$R_2=\frac{\gamma+1}{\gamma-1},\qquad P_2(\theta)=\frac{8}{25(\gamma+1)}\left[1+\left(\frac{\lambda_2'}{\lambda_2}\right)^2\right]^{-1} \qquad (5)$$

At the disturbed free surface

$$r_*(t,\theta)=\left(\frac{E}{\rho_1}\right)^{1/5}t^{2/5}\lambda_*(\theta) \qquad (6)$$

pressure is equal zero, and the particle velocity projections on the normal to the surface coincides with the normal velocity of the surface itself. These conditions may be written in the dimensionless form

$$P_*=0,\qquad U_*(\theta)-\frac{\lambda_*'}{\lambda_*}V_*(\theta)=\frac{2}{5} \qquad (7)$$

Subscript $*$ relates to magnitudes at the perturbed free surface. The problem is to find a solution of the system of Eqs. (1) satisfying conditions (5) and (7) along the unknown boundaries $\lambda_2=\lambda_2(\theta)$ and $\lambda_*=\lambda_*(\theta)$.

In order to solve the stated problem the motion in the neighborhood of the shock wave intersection point with the free surface (point B in Fig. 2). Passing as in [12] from variables λ and θ to a system of polar coordinates s and φ with its origin at point (Fig. 2) by Formulas

$$s = (\lambda^2 - 2\lambda\lambda_B \sin\theta + \lambda_B^2)^{1/2}, \qquad \varphi = \operatorname{arctg}\left(\operatorname{tg}\theta - \frac{\lambda_B}{\lambda\cos\theta}\right) \tag{8}$$

and expressing the looked for functions in the form of series expansions in s, we can obtain from system (1) for the limit values of the unknown functions $U_B(\varphi)$, $V_B(\varphi)$, $R_B(\varphi)$ and $P_B(\varphi)$ when $s = 0$ the following system of equations:

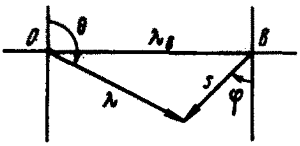

Fig. 2

$$R\left(U - \frac{2}{5} + V\operatorname{tg}\varphi\right)\frac{dU}{d\varphi} + \frac{dP}{d\varphi} = 0$$
$$R\left(U - \frac{2}{5} + V\operatorname{tg}\varphi\right)\frac{dV}{d\varphi} + \operatorname{tg}\varphi\frac{dP}{d\varphi} = 0$$
$$\left(U - \frac{2}{5} + V\operatorname{tg}\varphi\right)\frac{dR}{d\varphi} + R\left(\frac{dU}{d\varphi} + \operatorname{tg}\varphi\frac{dV}{d\varphi}\right) = 0 \tag{9}$$
$$\frac{dP}{d\varphi} - \frac{\gamma P}{R}\frac{dR}{d\varphi} = 0$$

Here and in the following subscript B is omitted. Point B is a singular point. The following boundary values of functions U, V, R and P when this point is approached along the shock wave and the free surface are defined here by

$$U_2 = \frac{4}{5(\gamma+1)}\cos^2\varphi_2, \qquad V_2 = \frac{4}{5(\gamma+1)}\sin\varphi_2\cos\varphi_2 \tag{10}$$
$$R_2 = \frac{\gamma+1}{\gamma-1}, \qquad P_2 = \frac{8}{25(\gamma+1)}\cos^2\varphi_2$$
$$P_* = 0, \quad U_* + \operatorname{tg}\varphi_* V_* = 2/5 \tag{11}$$

Here φ_2 and φ_* are the angles corresponding to the positions of the shock wave and the free surface at point B respectively. The question of finding a nontrivial solution of the system of Eqs. (9) at point B which would depend on angle φ and satisfy the boundary conditions may be considered.

It is readily seen that the system (9) has two algebraic integrals

$$P = KR^\gamma \ (K = \text{const}) \tag{12}$$
$$\gamma P / R = [(U - 2/5)\cos\varphi + V\sin\varphi]^2 \tag{13}$$

Integral (13) is the condition of nontrivial solvability of the system of Eqs. (9) with respect to derivatives.

With the use of (12), instead of (9) and (13) we can write the following system:

$$\frac{dU}{d\varphi} = \frac{2}{\gamma+1}\cos\varphi\left[\left(U - \frac{2}{5}\right)\sin\varphi - V\cos\varphi\right], \quad \frac{dV}{d\varphi} = \operatorname{tg}\varphi\frac{dU}{d\varphi} \tag{14}$$
$$R(\varphi) = \left\{\frac{[(U - 2/5)\cos\varphi + V\sin\varphi]^2}{\gamma K}\right\}^{1/(\gamma-1)} \tag{15}$$

A qualitative analysis of the flow in the vicinity of point B is given in detail in (12]. The point at which a rarefaction wave passes through the free surface is the point of origin of a rarefaction wave in which the pressure falls from its value at the shock front down to zero. A flow similar to the Prandtl-Mayer flow develops. The shock wave skims along the free surface at a velocity equal to $D / \cos\delta$ in which δ is the angle between the tangent to the shock wave at point B and the normal to the undisturbed free surface equal to angle β between the normal to the shock wave and the undisturbed free surface. If angle δ is sufficiently great, then the rarefaction waves cannot catch up with the shock front, and the envelope of the fore front of these waves will be the front of waves reflected from the free sutface. This type of reflections is called "regular". If however angle δ is small, the rarefaction waves catch up with the shock front, and by interaction distort the latter. In this case the reflected wave front is indeterminate ("irregular" reflection) at point B It follows from the considerations adduced in [12] that in the problem here considered the reflection is of a "critical" character. The equality of the velocity of shock wave motion along the free surface and the velocity of the rarefaction wave fore front is the condition of the "critical" character of reflection. The flow in this case, as in that of regular reflection consists of two regions, viz. the area between the shock wave ($\varphi = \varphi_2$) and the rarefaction wave fore front ($\varphi = \varphi_R$), where all of the flow characteristics are constant, and the area of the rarefaction waye itself ($\varphi_R \leqslant \varphi \leqslant \varphi_*$) in which pressure falls down to zero. The position of these two areas is shown in Fig. 3. The condition for the reflection to be "critical" has been inaccurately written in paper [12]. The correct expression for this condition is of the form

$$\gamma P_2 / R_2 = (U_2 - {}^2/_5)^2 + V_2^2 \tag{16}$$

where U_2, V_2, R_2 and P_2 are defined by relations (10). This condition may actually be obtained from the geometric construction shown in Fig. 4, where two consecutive positions of point B and of the fronts of the shock and rarefaction waves corresponding to

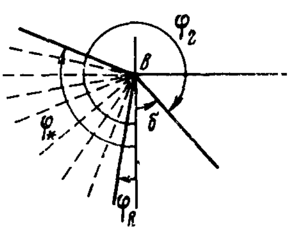

Fig. 3

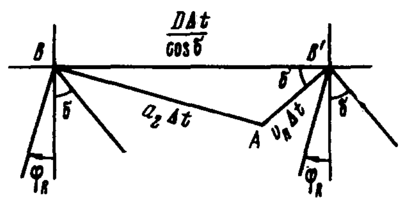

Fig. 4

instants t and $t + \Delta t$ are represented. The rarefaction wave front propagation may be represented schematically as propagation at the local velocity of sound a_2 along a normal to itself, followed by the transfer at velocity of particles behind the shock wave v_n in the direction of that velocity. In Fig. 4 this is represented by segments BA and AB' respectively. For the reflection to be critical the distances covered by the shock wave and the rarefaction wave front along the free surface must be equal to the time interval Δt. During the time interval Δt the shock wave would have travelled along the free surface a distance equal to $D\Delta t / \cos\delta$. For triangle BAB' we obtain the relation

$$a_2^2 = (D \sec\delta)^2 + u_2^2 + v_2^2 - 2D \sec\delta \sqrt{u_2^2 + v_2^2} \cos\delta \tag{17}$$

Here $\sqrt{u_2^2 + v_2^2} = |v_n|$ is the absolute value of particle velocity behind the shock wave. Obviously

$$\sqrt{u_2^2 + v_2^2} \cos\delta = u_2$$

Passing to dimensionless variables we obtain condition (16). The same condition may be derived from the algebraic integral (13). For $\varphi = \varphi_R$ we have

$$\gamma P_2 / R_2 = [(U_2 - {}^2/_5) \cos\varphi_R + V_2 \sin\varphi_R]^2$$

Solving this equation for φ_R we obtain

$$\varphi_R = -\arcsin \frac{\sqrt{\gamma P_2 / R_2} V_2 + (U_2 - {}^2/_5) \sqrt{(U_2 - {}^2/_5)^2 + V_2^2 - \gamma P_2 / R_2}}{(U_2 - {}^2/_5)^2 + V_2^2} \tag{18}$$

It will be seen from (18) that the transition from regular reflection to an irregular one is determined by condition (16) from which follows

$$\varphi_{2*} = -\arc\cos\sqrt{{}^1/_2(\gamma+1)/\gamma}$$

Here φ_{2*} is the angle corresponding to the shock wave position at point B for a critical reflection.

Numerical integration of the system of Eqs. (14) with boundary conditions (10) and (11), and also with the condition of the solution continuity at the boundary $\varphi = \varphi_R$ was carried out with $\gamma = 7$. The supplementary condition (16) was necessary because in this case the two conditions (11) coincide identically, as is seen from (11), (12) and (15). Computation results are presented in Table 1.

Table 1

$\varphi°$	U	V	R	P	U^*	V^*
$\varphi = \varphi_R = 9°22'$	0.05713	−0.04949	1.3333	0.02285	0.05713	−0.04949
19°38′	0.05549	−0.04995	1.3268	0.02207	0.05549	−0.05066
31°44′	0.05057	−0.05239	1.3050	0.01966	0.05054	−0.05357
41°16′	0.04506	−0.05651	1.2767	0.01686	0.04497	−0.05808
52°16′	0.03787	−0.06422	1.2308	0.01305	0.03768	−0.06624
64°00′	0.03052	−0.07612	1.1639	0.00883	0.03018	−0.07890
74°16′	0.02548	−0.08943	1.0869	0.00547	0.02500	−0.09294
83°48′	0.02272	−0.10373	0.9948	0.00294	0.02216	−0.10804
96°16′	0.02276	−0.12422	0.8258	0.00080	0.02220	−0.12973
106°32′	0.02625	−0.14151	0.5853	0.00007	0.02584	−0.14812
$\varphi = \varphi_* = 112°24'$	0.02966	−0.15115	0.2120	0.00000	0.02944	−0.15838

The position of the boundary $\varphi = \varphi_*$ is of fundamental significance. Contrary to [12] this boundary lies above the level of the undisturbed free surface. Theoretical and experimental investigations show (see, e. g. [9–12]) that when an explosion occurs at the half-space boundary, the flow cannot be confined to one side only of the undisturbed free surface. In the context of the formulation used here this can also be proved by the analysis of relations (10)–(14). The representation of the free surface form proposed in [12] cannot be used directly, if the proper position of the free surface is taken into consideration. The disposition of boundaries φ_2, φ_R and φ_* occurring in a shock wave critical reflection from the free surface is shown in Fig. 3.

Solution of the system of Eqs. (14) may be presented in the form of expansions as follows:

$$U(\varphi) = U_0(\varphi) + \frac{U_1(\varphi)}{\gamma+1} + \ldots + \frac{U_n(\varphi)}{(\gamma+1)^n} + \ldots$$
$$V(\varphi) = V_0(\varphi) + \frac{V_1(\varphi)}{\gamma+1} + \ldots + \frac{V_n(\varphi)}{(\gamma+1)^n} + \ldots \quad (19)$$

Substituting these expansions into Eqs. (14) and into the boundary conditions we obtain for $\varphi = \varphi_R$

$$dU_0/d\varphi = dV_0/d\varphi = 0, \quad U_0(\varphi_R) = V_0(\varphi_R) = 0$$
$$dU_1/d\varphi = -{}^2/_5 \sin 2\varphi, \quad U_1(\varphi_R) = {}^4/_5 \cos^2\varphi_2$$
$$dV_1/d\varphi = {}^{\varphi}\cdot {}^4/_5 \sin^2\varphi, \quad V_1(\varphi_R) = {}^2/_5 \sin 2\varphi_2$$
$$dU_n/d\varphi = U_{n-1}\sin 2\varphi - 2V_{n-1}\cos^2\varphi, \quad U_n(\varphi_R) = 0$$
$$dV_n/d\varphi = 2U_{n-1}\sin^2\varphi - V_{n-1}\sin 2\varphi, \quad V_n(\varphi_R) = 0 \quad (n = 2, 3, \ldots) \quad (20)$$

Very rough estimates show that U_n and V_n increase with increasing angle φ at a rate not exceeding φ^n, hence series (19) will be uniformly convergent, at least for $|\varphi| \leqslant \varphi' < \gamma + 1$, where $\varphi' > 0$ is any given value of angle φ smaller than $\gamma + 1$. Equations (19) define in this region the continuous differentiable solution of the system of Eqs. (14).

From (20) the following expressions of coefficients of the first three expansion terms may be derived

$$U_0 = V_0 = 0$$

$$U_1 = {}^1/_5 \cos 2\varphi + C_{11}, \quad C_{11} = {}^4/_5 \cos^2 \varphi_2 - {}^1/_5 \cos 2\varphi_R$$

$$V_1 = {}^1/_5 \sin 2\varphi - {}^2/_5 \varphi + C_{12}, \quad C_{12} = {}^2/_5 \sin 2\varphi_2 - {}^1/_5 \sin 2\varphi_R + {}^2/_5 \varphi_R$$

$$U_2 = f(\varphi) + C_{21}, C_{21} = -f(\varphi_R); \; V_2 = g(\varphi) + C_{22}, \; C_{22} = -g(\varphi_R) \tag{21}$$

$$f(\varphi) = -{}^1/_2 C_{12} \sin 2\varphi - {}^1/_2 (C_{11} - {}^2/_5) \cos 2\varphi + \varphi({}^1/_5 \sin 2\varphi - C_{12}) + {}^1/_5 \varphi^2$$

$$g(\varphi) = {}^1/_2 C_{11} \sin 2\varphi + {}^1/_2 C_{12} \cos 2\varphi - \varphi({}^1/_5 \cos 2\varphi + C_{11} - {}^1/_5)$$

Values of $U(\varphi)$ and $V(\varphi)$ denoted by asterisks obtained for the same values of angle φ are given in Table 1. Although coefficients (21) of the first three terms only of expansions (19) were used in computations, the deviation from exact values does not exceed 5%.

The value of φ_* calculated with the use of the first two terms only of expansions (19) is $\varphi_* = 113°29'$, while with the third term taken into account it was $\varphi_* = 113°26'$.

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